Robust Risk Estimation and Hedging: A Reverse Stress Testing Approach

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ABSTRACT

Traditional risk modeling using Value-at-Risk (VaR) is widely viewed as ill equipped for dealing with tail risks. As a result, scenario-based portfolio stress testing is increasingly being promoted as central to the risk management process. A recent innovation in portfolio stress testing endorsed by regulators, called reverse stress testing, is intended to identify economic scenarios that will threaten a financial firm's viability, but do so without injecting the manager's cognitive biases into stress scenario specification. While the idea is intuitively appealing, no template has been provided to operationalize the idea. Some first steps in developing reverse stress testing approaches have begun to appear in the literature. Complexity and computational intensity appear to be important issues. A more subtle issue appearing in this emerging research is the relationship among the concepts of likelihood, plausibility, and representativeness. In this paper, we propose a novel method for reverse stress testing. The process starts with a multivariate normal distribution and uses Principal Components Analysis (PCA) along with Gram-Schmidt orthogonalization to determine scenarios leading to a specified loss level. The approach is computationally efficient. The method includes the maximum likelihood scenario, maximizes (a definition of) representativeness of the scenarios chosen, and measures the plausibility of each scenario. In addition, empirical results for sample portfolios show this method can provide new information beyond VaR and standard stress testing analyses.

INTRODUCTION

Stress testing's aim is to elucidate the level of portfolio loss under the condition that a specified event occurs (i.e., a conditional loss forecast). This contrasts with the risk measure known as Value-at-Risk, which defines a level of portfolio loss expected to be exceeded with a specified probability (a quantile of a forecast loss distribution of a specific form). Over the last fifteen years, use of stress testing has gained ever-wider currency, fueled by perceived failings of Value-at-Risk and other traditional risk models under extreme events.[1]

[1]See, e.g., Jorion (2001, 2006).

Stress Testing is thought to complement traditional risk models by focusing on events that are not represented in traditional risk forecasts, either because they are absent from or under-represented in the historical record. Thus a stress test, unlike a quantile forecast, it is not defined in relation to all possible states of the world and their estimated probabilities. This claim may be overstated, however, as stress testing covariance matrices may actually confound the conditional loss forecast with the statistical density forecast.[2]

In addition to the conceptual problems just alluded to, stress testing also entails design obstacles as a risk forecasting technique. One of the most troublesome is the subjective nature of the specified shocks. Indeed, this is especially important in cases in which shocks are not explicitly set by regulators (which is common only in the banking industry), e.g., under company-generated stress scenarios in the US, as required under the Dodd-Frank Act. It is not easy to demonstrate that a particular stress testing scenario, specifically the magnitude of the various stress shocks, have been chosen unbiasedly and represent risks relevant to the financial firm's decision making. In other words, plausibility and relevance must be demonstrated, with the emphasis on plausibility. Reverse stress testing has as its main motivation the goal of overcoming this particular objection.

REVERSE STRESS TESTING

The Federal Reserve Board has described reverse stress testing as a process in which banks first "assume a known adverse outcome… then deduce the types of events that could lead to such an outcome." (Federal Reserve Board (2012), p. 12). In other words, instead of asking 'what will happen to my portfolio if oil rises X % or S&P 500 drops Y %?'; a user performing a reverse stress test will simply ask 'what are the plausible ways for my portfolio to lose Z %?'. Reverse stress testing has a short history. First suggested in the report of the Counterparty Risk Management Policy Group III (2008), it has been embraced by the banking regulatory community, perhaps most enthusiastically by the UK's Financial Securities Authority (2008)

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[2] See Novosyolov and Satchkov (2010). Discussion of those problems is outside of the scope of this paper, but it is important to note that any reverse stress testing will be subject to the same issues. Thus, the distribution that serves as the foundation for reverse stress tests must be built to reflect the actual tail behavior, not the extrapolated normal behavior. With that note, we will focus on mechanics of the reverse stress testing and will leave the distributional issue for a separate research study.

(See also Basel Committee on Banking Supervision (2009), and above-mentioned Federal Reserve Board (2012). Absent from the regulatory endorsements is any indication of how such scenarios are to be identified computationally. For a (non-degenerate) multi-asset portfolio an infinite number of solutions to this question are possible.

Several authors have recently proposed approaches to the identification of meaningful reverse stress tests. See, Breuer and Csiszar (2010), Breuer, Jandacka, Rheinberger, and Summer (2009), Flood and Korenko (2010), Glasserman, Kang, and Kang (2012), Grundke (2011, 2012), and Skoglund and Chen (2009). Most of these papers share the common theme of identifying the maximum likelihood scenario as a way of selecting one scenario as most relevant.[3] However, identifying a single scenario as relevant may be an approach that too optimistically relies on the data used to parameterize the return distribution. Mirzai and Müller (2013) use an approach based on a large number of portfolio simulations and a heuristic method for identifying the characteristics of meaningful scenarios.

The approach in this paper differs from other work in that it selects multiple scenarios in a systematic fashion so that they are plausible and maximize the differences from one another under a specific measure. One of the scenarios selected is always the maximum probability scenario corresponding to the given loss level. The methodology is discussed next.

IDENTIFYING SCENARIOS UNDER REVERSE STRESS TESTING

The first and a deceptively easy way to identify scenarios is to simply use standard risk decomposition based on Euler's theorem for homogenous functions. In the case of VaR, it is commonly known as Component VaR. One could take the Z% loss specified above and simply examine Component VaR[4] that corresponds to that loss level. Obviously, that would only be one scenario and it is what all risk managers currently use[5]. However, it is important to keep that decomposition in mind, since we will later show that this standard risk decomposition will

[3] McNeil and Smith (2011), examining a related problem, show that the maximum probability scenario for a given level of loss is isomorphic to the Value-at-Risk for some confidence level α (under certain regularity conditions).
[4] Sometimes this term is called Incremental VaR, though we prefer to reserve the use of that term for the partial duration of the risk measure with respect to weight. Component VaR should add up to total VaR (this could also be called a contribution to tracking error, since in a parametric world the two are equivalent).
[5] Recall, that we left the differences in modeling distributions outside of the scope of our discussion here.

correspond to the most likely scenario in any set of reverse stress tests over a given loss hyperplane. Other ways of losing Z % could be plausible, even likely, but the one corresponding to the Euler's decomposition is actually a scenario with most likelihood (highest density conditional on the event that the loss of Z % occurs). We will later put forth the proof of this assertion, but it is worth considering this Component VaR, since it will be the starting point for our thinking on the topic. But even the fact that a scenario has the highest density certainly does not mean that it is the only scenario of interest or that focusing on this most likely scenario is productive for risk management. For more on this, please see the Discussion of Results section.

Before we employ any mathematical machinery we must understand conceptually what kind of scenarios we are looking for when we are performing reverse stress testing. We would posit that these scenarios must satisfy the following criteria:

a. They are likely

b. They are at least somewhat different (otherwise we could just do with one scenario from our standard risk decomposition to describe it all)

c. They are not missing any danger scenarios

Now that we have some specification of what we are looking for, let us examine the tools that will be necessary.

REVERSE STRESS TESTING IMPLEMENTATION

Given an asset vector $X = (X_1, \ldots, X_n)'$ possessing multivariate normal distribution with zero mean and covariance matrix $D = \mathbf{E}(XX')$. Consider portfolio weights $w = (w_1, \ldots, w_n)'$ and portfolio return $p = w'X$. The latter possesses univariate normal distribution with zero mean and variance $w'Dw$.

Next, given a loss level (say, VaR at a specified confidence level) L, which scenarios would lead to such a loss? The complete answer to this question is: these are scenarios $x \in R^n$ satisfying the equation

$$w'x = L. \quad (1)$$

Now we want to select a few scenarios from this whole set of them. Denote H the hyperplane in R^n defined by the equation (1), that is $H = H(w, L) = \{x : w'x = L\}$. The conditional distribution of X given $X \in H$ is clearly normal with mean a_H, let us denote it by $N(w, L)$.

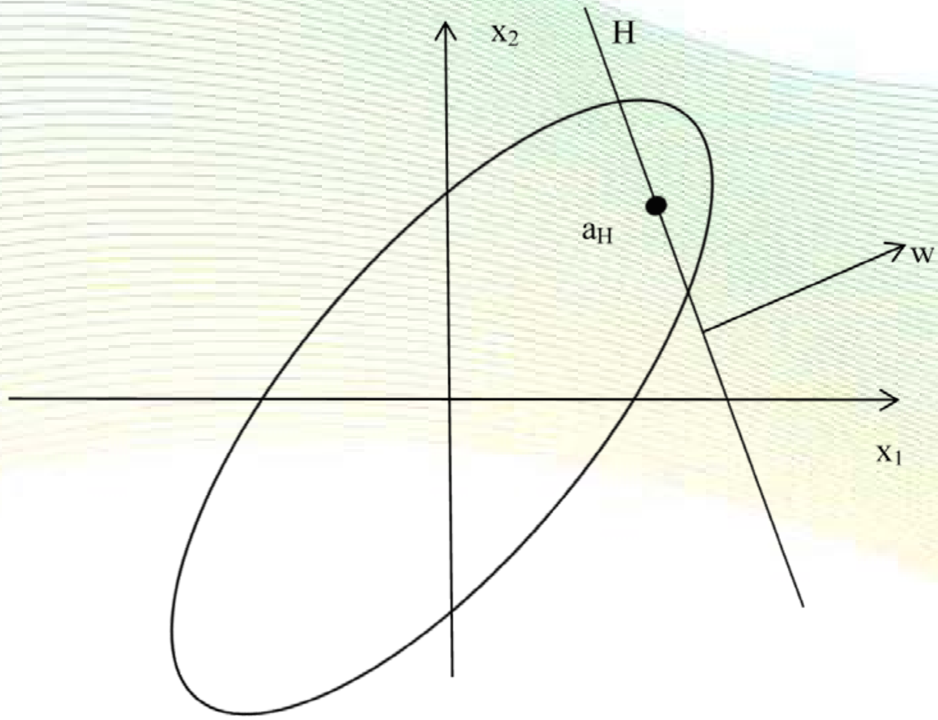

Figure 1. Initial distribution ellipse, portfolio hyperplane, and the mean of the conditional distribution $N(w, L)$.

The vector a_H is calculated as follows. The point a_H is such that some ellipsoid $x'D^{-1}x = const$ is tangent to the hyperplane Hat the point a_H, so the gradient of $x'D^{-1}x$ is proportional to w, thus

$$\nabla(x'D^{-1}x)\big|_{x=a_H} = 2D^{-1}x\big|_{x=a_H} = 2\beta w,$$

or $a_H = \beta Dw$. On the other hand, (1) implies that $w'a_H = L$, thus $w'\beta Dw = L$ which gives $\beta = L/(w'Dw)$ and

$$a_H = \frac{LDw}{w'Dw}. \tag{2}$$

In fact, the vector a_H also marks the maximum of conditional distribution density over H. Indeed, the conditional distribution density is proportional to the unconditional one, thus maximizing the density is equivalent to minimizing $x'D^{-1}x$ over H. Necessary and sufficient optimality condition for quadratic programming problem states that the gradient of the goal function should be orthogonal to H. In other words, an ellipsoid $x'D^{-1}x = const$ should be tangent to H at the optimal point. These are exactly the conditions that led to (2).

As noted above, the vector a_H coincides with the gradient of Value-at-risk in component VaR decomposition. Indeed, Value-at-risk at confidence level α is $v_\alpha = q_\alpha\sqrt{w'Dw}$, where q_α is the quantile of the standard normal distribution at level α. Its gradient is

$$\nabla v_\alpha = q_\alpha \frac{Dw}{\sqrt{w'Dw}}.$$

Given loss $L = v_\alpha$, we have $q_\alpha = L/\sqrt{w'Dw}$, so

$$\nabla v_\alpha = \frac{LDw}{w'Dw}, \tag{3}$$

which coincides with (2).

We will describe the process of building desired scenarios in two directions. First we will make some transforms to a convenient position, from which our algorithm can do its work, and then simply use inverse transforms to get back to the familiar coordinate plane.

DIRECT TRANSFORM.

First step of direct transform is intended to make the description of conditional normal distribution on a hyperplane more convenient. To achieve this we look for a pivot that moves weights vector w to the vector $w_Y = (\|w\|_2, 0, \ldots, 0)'$. This may be done as follows. Start with a matrix of full rank with w as a first row, and apply the Gram – Schmidt orthogonalization procedure to its rows, denote the resulting orthogonal matrix P. One can easily see that indeed $Pw = w_Y$. The asset vector X is transformed by this matrix to the random vector

$$Y = PX \tag{4}$$

with zero mean and covariance matrix

$$D_Y = PDP'.$$

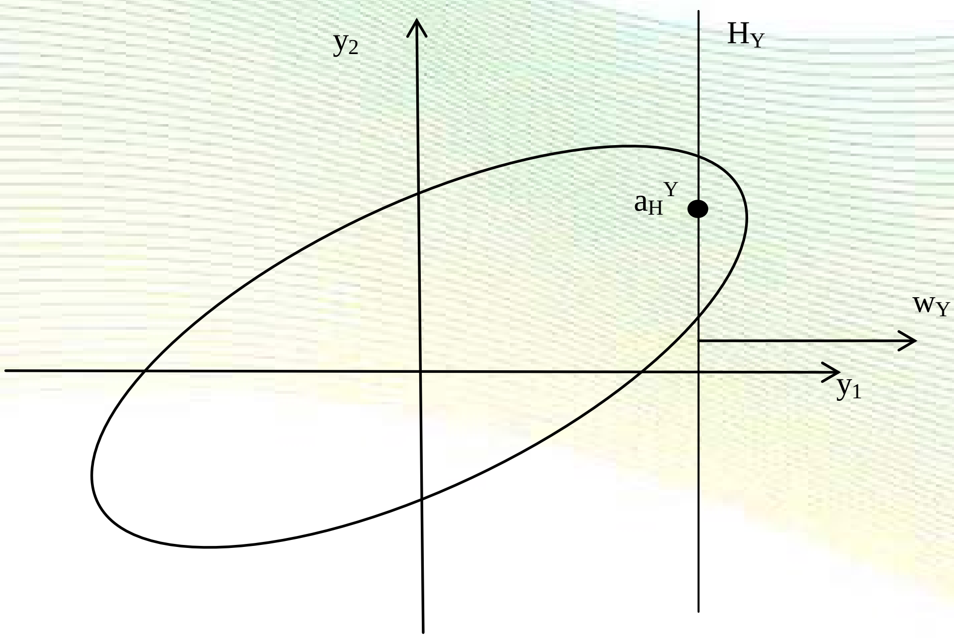

Figure 2. The distribution an portfolio parameters after pivoting using the matrix P

The hyperplane H goes to another hyperplane $H_Y = PH$ defined by

$$H_Y = \{y : \ w_Y' y = L\}.$$

By the appearance one can see that for $y \in H_Y$ we have $y_1 = L/\|w\|_2$, and other coordinates make take any values. This means that H_Y is parallel to the coordinate hyperplane, making it easy to describe the conditional distribution of Y given $Y \in H_Y$. Indeed, denote $I = \{2, \ldots, n\}$, and consider a block structure of the covariance matrix D_Y of the form

$$D_Y = \begin{pmatrix} d_{11} & D_{1I} \\ D_{I1} & d_{II} \end{pmatrix}.$$

It is well known that the distribution of $(Y_2, \ldots, Y_n)$ given $Y_1 = L/\|w\|_2$ is normal with mean a_Y^H

and covariance matrix D_Y^H

$$a_H^Y = \frac{L}{d_{11}\|w\|_2} D_{I1}, \quad D_Y^H = D_{II} - \frac{1}{d_{11}} D_{I1} D_{1I} \tag{5}$$

Note, that the only reason for the above gymnastics is mathematical convenience; conceptually it is not necessary. We simply had to rewrite a degenerate distribution, because we took away one of its degrees of freedom by constraining a portfolio to a specific loss, but did not remove any corresponding variable in the parametrization. Now that the corresponding variable is removed, we have a mathematically tractable distribution and we can move on with finding our scenarios. After we are done, we will pivot it back to the original distribution to analyze the results.

EQUIDISTANT POINTS.

The second step is also a mathematical device, but it does not have any relation to the initial distribution. It simply provides $M+1$ points, one of them being the origin, and the rest M would lie inside the ball of a given radius R as far from each other as possible. For $M \leq n$ the latter M points are actually vertices of a regular $(M-1)$ - simplex inscribed into the ball, e.g. for $M=3$ this is a regular inscribed triangle. The M points in R^{M-1} may be found as a solution to the optimization problem

$$G(x^1, \ldots, x^M) = \sum_{j=0}^{M-1} \sum_{k=j+1}^{M} \frac{1}{d(x^j, x^k)} \to \min_{x^1,\ldots,x^M} \tag{6}$$

subject to

$$d(x^j, x^0) \leq 1, \quad j = 1, \ldots, M \tag{7}$$

where $d(x,y)$ stands for Euclidean distance between x and y

$$d(x,y) = \sqrt{\sum_{i=1}^{M-1} (x_i - y_i)^2}\ ,$$

and x^0 stands for the origin. Note that the problem (6), (7) has infinitely many solutions, which may be obtained from one another by pivoting the ball (7) around the origin. This is illustrated in figure 3.

After the solution we just add $n-M$ zeros to the end of each vector $x^0, x^1, \ldots, x^M$ to turn them into R^{n-1} vectors.

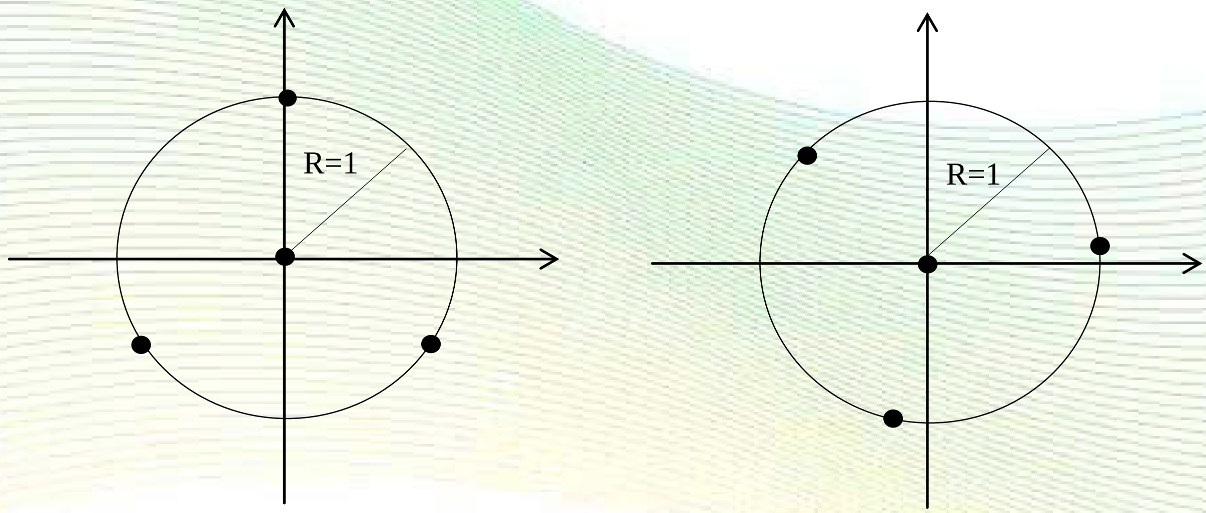

Figure 3. Two possible solutions of the problem (6), (7) for $M=3$

To fix the problem we require that the first point x^1 has exactly one (first) non-zero component, so that inevitably $x_1^1 = 1$, the second point has only two first non-zero components, so that $x_3^2 = \cdots = x_{M-1}^2 = 0$, and so on,

$$x_i^j = 0; \; i = j+1, \ldots, M-1; \; j = 1, \ldots, M-2. \tag{8}$$

This would ensure that after going to principal components space first points always reside within first principal components subspace, thus possessing the largest possible likelihood.

ADJUSTING TO PRINCIPAL COMPONENTS.

The third step is moving the selected $M+1$ points in such a way that they would represent the most likely scenarios among those located at given distances from each other. To achieve this denote V the $(n-1)\times(n-1)$ matrix whose columns are eigenvectors of D_Y^H, sorted in descending order of eigenvalues, and apply the transform

$$z^j = V' x^j, \;\; j = 0, 1, \ldots, M.$$

After that calculate radius so that the minimum relative likelihood (defined later) equals to the required value q, which gives

$$R = \sqrt{-\frac{\ln q}{z^{M\prime}\left(D_Y^H\right)^{-1} z^M}}, \tag{9}$$

and finalize the transform by

$$y^j = Rz^j + a_H^Y, \quad j = 0, 1, \ldots, M.$$

Note that since only first M components of each x are nonzero, the y's populate the most likely area of the hyperplane H_Y, as desired. Here we also calculate the values of conditional density function f at each scenario $y^j,\ j = 1, \ldots, M$, relative to the value of density at y^0:

$$k_j = \frac{f(y^j)}{f(y^0)}, \quad j = 1, \ldots, M. \tag{10}$$

Let us illustrate adjusting to principal components by a figure.

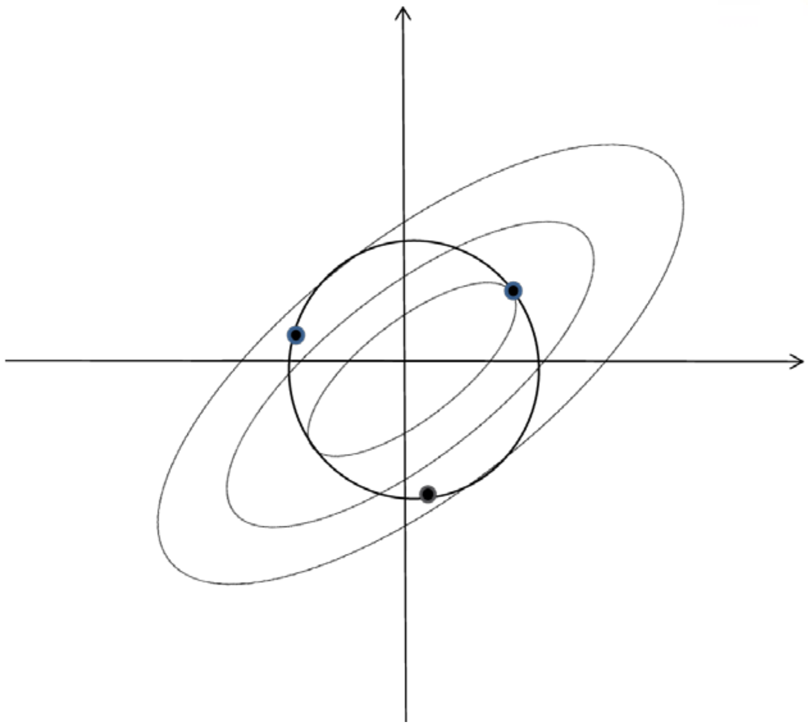

Figure 4. Adjusting to first principal components

BACK TO INITIAL PLACEMENT.

To finalize we should pivot H_Y back to H, which is easily done using the P' matrix. First add to the beginning of each vector y the constant first component $L/\|w\|_2$ to turn them to n -dimensional vectors, and finally apply the pivot

$$X^j = P'y^j, \quad j = 0, 1, \ldots, M.$$

These are the desired scenarios, the result of reverse stress testing. They may be ranked with respect to values of relative likelihood calculated in (10).

ADDITIONAL NOTES

The *number of scenarios* taken may be tied to variance explained by principal components. E.g. if we'd like to stay within principal components explaining 80% of variance, and this is achieved by 5 principal components then we should constrain ourselves with 6 scenarios which would fit into the span of the first 5 principal components. In practice the number of scenarios may also be selected from convenience point of view, thus $M = 5$ seems an appropriate choice for most cases.

The relative likelihood q may be chosen as 0.1, meaning that the less likely scenario possesses the relative likelihood 10 times less than the central scenario a_H.

HOLDINGS BASED MODEL

In this case we have a model

$$r = \alpha + G\tilde{X} + R \tag{11}$$

where r stands for securities returns, α denotes risk-free rate, G means loadings to factors $\tilde{X}$, and R stands for residuals (idiosyncratic risk). Here r, R, α are $m \times 1$ vectors, $\tilde{X}$ is an $n \times 1$ vector, and the loadings matrix G has the size $m \times n$. Denote $\tilde{D}$ the covariance matrix of $\tilde{X}$. For the purpose of reverse stress testing we will ignore the residual risk by setting $R = 0$.

Now consider a portfolio of securities with weights $\tilde{w}$ so that the portfolio return is

$$P = \tilde{w}'r = \tilde{w}'\alpha + \tilde{w}'G\tilde{X}.$$

Given a portfolio shock $\tilde{L}$, denote $L = \tilde{L} - \tilde{w}'\alpha$, and $\overline{w}' = \tilde{w}'G$, thus obtaining a factors portfolio representation

$$\overline{w}'\tilde{X} = L. \tag{12}$$

This latter representation has a disadvantage that some factors possess huge variance and small loadings, which result in small weights in $\overline{w}$. These factors do not bring much value to portfolio, but dominate other factors in principal components decomposition, which is misleading. To overcome the trouble, we rescale the problem in such a way that weights in $\overline{w}$ have similar order of magnitude.

To achieve this, first eliminate zero and insignificant weights in $\overline{w}$. Choose a small $\varepsilon > 0$ and denote

$$J = \{j \in \{1, \ldots, n\} : \ |\overline{w}_j| > \varepsilon\}.$$

Later on we will keep only weights (and factors) with indices in J, setting other weights to 0. Denote $N = |J|$ the number of elements in J, and introduce multipliers $v_j = N\overline{w}_j,\ j \in J$. Now we let

$$w_j = \frac{\overline{w}_j}{v_j} = \frac{1}{N},\ j \in J$$

and

$$X_j = v_j\tilde{X}_j,\ j \in J.$$

Next let us collect the components $X_j,\ j \in J$ into the column vector X, and the components $w_j,\ j \in J$ into the column vector w. The covariance matrix D of X is obtained from $\tilde{D}$ by selecting columns and rows with numbers in J and scaling both by the same multipliers. More formally, denote V the diagonal $N \times N$ matrix with numbers $v_j,\ j \in J$ in its diagonal, and $\tilde{D}_{JJ}$ the selected rows and columns of $\tilde{D}$, then

$$D = V\tilde{D}_{JJ}V. \tag{13}$$

We reduced the problem to finding scenarios such that

$$w'X = L. \tag{14}$$

Now apply the standard procedure to build $m+1$ scenarios $x^k,\ k=0,\dots,m$, and convert them to initial scaling by scaling components of x's with v

$$\tilde{x}_j^k = x_j^k/v_j,\ j \in J,\ k=0,1,\dots,m,$$

and filling the rest places of scenarios with zeros.

DISCUSSION AND APPLICATIONS

Next, we will discuss two applications for the algorithm described above:

1. Enhance standard decomposition of tracking error or VaR that is typically based on Euler's theorem for homogenous functions by adding additional plausible scenarios. Form quasi-confidence intervals for contributions to TE/VaR to report the contributions that are stable in various scenarios at the same loss level.
2. Suggest efficient intra and inter-asset class hedging strategies at a given loss level.

Application 1: Enhance risk decomposition techniques by adding plausible scenarios beyond the standard decomposition based on Euler's theorem for homogenous functions

As we have already shown in formula (2), the center of our spherea_H coincides of the gradient of VaR with respect to the vector of portfolio weights. It is well known that this gradient serves as the basis for traditional risk decomposition based on the Euler theorem, which states that for any homogenous function of degree 1:

$$f(x) = \sum_i x_i \frac{\partial f(x)}{\partial x_i} \tag{15}$$

Since parametric VaR function has the required properties, the total risk budget is apportioned using the Component VaR equal to the right hand side of the equation (15), see Jorion (2001). Thus, the decomposition in formula (15) is currently the primary means for quantifying contributions from asset class, manager, desk or asset contributions to the overall risk budget. Crucial decisions from strategic asset allocation to compensation are taking into account this specific form of the risk budget. And while, as we have shown, the Component VaR based on the Euler theorem is the outcome associated with the highest conditional density over a given loss hyperplane, it is still only one outcome. If there were other outcomes that are still likely (even if that likelihood is below maximum by definition) and different enough, it would be very important for any risk manager to be aware of them. Consider the table in Exhibit 1. In it we have an equal weighted portfolio of a thirteen various asset classes and we are examining the scenarios that can lead to the loss level of -15%. Each title contains a description of the form Scen A: X (of Y%). The "A" is the number of the scenarios; we chose up to five scenarios in addition to scenario zero, which corresponds to point a_H. The value "X" is calculated by the formula (10) and represents the likelihood (plausibility) of a scenario relative to the most conditionally likely scenario zero. Finally, "Y" represents the inverse VaR that corresponds to this loss level. In a table below this value is equal to two percent, which means that a loss level of -15% corresponds to a 98% VaR for this portfolio.

Exhibit 1 – Contribution to loss from equidistant points on the sphere in the loss hyperplane

Name	Weight %	Scen 0 : 100% (of 2%)	Scen 1 : 52% (of 2%)	Scen 2 :12% (of 2%)	Scen 3 :3% (of 2%)	Scen 4 : 1% (of 2%)	Scen 5 :1% (of 2%)	Contr. Stability
EUR	7.69	-2.47	-2.31	-2.88	-1.79	-2.79	-2.59	6.88
SP500 Health Care	7.69	-1.36	-1.74	-1.63	-1.37	-0.98	-1.07	4.99
AUD	7.69	-2.82	-3.34	-3.25	-1.79	-3.59	-2.11	4.28
Russell Equity (Russell 2000)	7.69	-2.04	-2.80	-1.88	-1.88	-2.40	-1.22	4.16
CHF	7.69	-1.53	-1.23	-1.82	-0.90	-1.64	-2.08	3.97
TIMBER	7.69	-1.79	-2.36	-1.70	-1.78	-2.26	-0.85	3.65
Germany Equity (Frankfurt XetraDax)	7.69	-1.97	-2.98	-1.78	-1.52	-2.45	-1.13	3.27
Gold Commodity	7.69	-1.16	-0.56	-1.15	-0.87	-1.23	-1.98	2.66
GBP	7.69	-1.47	-1.54	-1.44	-0.50	-2.43	-1.42	2.62
Germany Bond (10 GOV TR)	7.69	0.70	1.29	1.45	-0.69	1.68	-0.25	0.79
Industrials AAA	7.69	0.20	0.48	0.21	-0.42	0.42	0.33	0.68
JPY	7.69	0.33	0.79	0.03	0.09	1.17	-0.44	0.63
US Bond (7-10 GOV TR)	7.69	0.38	1.29	0.85	-1.58	1.49	-0.18	0.36
Grand Total	100	-15	-15	-15	-15	-15	-15	NA

It is plain to see from Exhibit 1 that traditional VaR decomposition would have left out significant additional information about the risk profile of the portfolio. We have added a measure called Contribution Stability to gauge the degree of reliability provided by the decomposition based on the Euler's theorem for homogenous functions. It is calculated as the absolute result of the ratio of scenario zero divided by the standard deviation of all chosen scenarios. This is just one of the alternatives that could be used to gauge the stability. At the top of the list is a EUR position. Its contribution at the 15% loss level is relatively stable, ranging only from -1.79 to -2.79. At the opposite end is a US Treasury position which can give a 1.58% loss of 1.49% gain. Clearly, scenario zero contribution of -2.47% (equivalent to standard Component VaR) for the EUR position is reasonably representative of its loss potential. However, scenario zero contribution of .38% does not paint an accurate picture of the loss potential of the US Treasury across different plausible environments. Thus, additional scenarios provide useful information about the reliability of the standard Component VaR for any given position and show the potential for plausible deviations from it.

Application 2: Suggest efficient hedges

However, the applicability of the algorithm does not stop there and can be extended in a number of directions, one of which is hedging. Hedging is done in a variety of contexts. When a specific risk to be hedged is known in advance with certainty, hedging then reduces to finding an instrument that has a transparent relationship to this risk, calculating the necessary hedging ratio and implementing the hedge. Let's call this the 'simple' hedging. However, not all hedging is reduced to such neat algorithm. Frequently, hedging will mean overall de-risking of a complex portfolio which has a very complex set of interrelated exposures e.g. a multi-asset class portfolio that is widely diversified. Rather than discussing various possibilities for de-risking of the portfolio, we can try to imagine an ideal instrument that would possess the same properties as an instrument used in our 'simple' hedging process. This would be like creating a derivative based on the actual portfolio held by the client. But where would we find such an instrument and what would be its cost? The Reverse ST process outlined above suggests that we can come close at a very low cost, by finding asset classes/factors/instruments that are closely related to our portfolio in all scenarios and for which the magnitude of moves can be reasonably closely linked to a given loss level in a portfolio. The key to this process is the observation that in the Reverse ST algorithm we are not limited to observing the behavior of asset classes/assets/factors that are held in a portfolio. We can include any number of factors and understand how they relate to our portfolio when our portfolio incurs a specific loss. Among those we can look for our ideal hedge of a basket of hedges. Such a hedge could possess three great properties:

a. We can find assets that move in line with our portfolio, but with bigger scale. This will give us a chance to buy inexpensive insurance, because we would only be looking for a payoff given a large move in the underlying.
b. We can find assets that are related to portfolio in many different scenarios and not just in a most likely one.
c. We can look across wide universe of available assets/asset classes/factors to find tradable instruments for which derivatives are readily available.

Let's consider Exhibit 2. The scenarios shown in it are exactly the same as the ones in Exhibit 1. There are two differences, however. The first is that the list of asset classes is extended to include asset classes not held in a portfolio. The second is that returns are not in the contribution form, but rather in a standalone form, that is, not scaled by the weight. This return can be interpreted as the return corresponding to a given plausible scenario tailored such that a given portfolio loses 15%. The list in Exhibit 2 is only partial and in theory there are no limits as to how many asset classes can be examined[6].

By analogy with Contribution Stability we calculated a "Hedge Robustness" (HR) measure, which has the same formula (a stylized Z-score), but is calculated on standalone returns without regard for portfolio weightings. The higher the HR is, the higher the scenario zero absolute return is, thereby satisfying the property (a) above. Also, the higher the HR is, the less variability it exhibits among the chosen scenarios i.e. property (b) above. Property (c) is achieved by including many different asset classes for which we can readily find derivative instruments. The top hedge in a list is EUR, which is not surprising given the analysis of Exhibit 1. However, EUR is already held in a portfolio and we may assume that this exposure is desired on other grounds. The following best hedges are Argentina Equity, Oil, Spain Equity, Russell Growth, AUD. Any one of them or a basket of them will serve as a good hedge for the portfolio. The

[6] Rather, the limits will come from sources outside of the Reverse ST algorithm, like stability of the covariance matrix estimators.

hedging ratios are easy to calibrate given the chosen loss level and the corresponding loss levels of each asset class in question. This allows for the choice of various deep out-of-the-money liquid derivatives for de-risking, which is far less costly than the OTC hedges currently placed on such portfolios.

Exhibit 2 – Standalone returns for various asset classes and Hedge Robustness

Name	Weight %	Scen 0 : 100% (of 2%)	Scen 1 : 52% (of 2%)	Scen 2 : 12% (of 2%)	Scen 3 : 3% (of 2%)	Scen 4 : 1% (of 2%)	Scen 5 : 1% (of 2%)	Hedge Robustness
EUR	7.7	-32.1	-30.0	-37.5	-23.3	-36.3	-33.7	6.9
Argentina Equity	0.0	-42.5	-56.4	-44.7	-39.7	-42.2	-29.7	5.4
GENERIC 1ST CRUDE OIL, WTI	0.0	-34.7	-45.7	-33.2	-26.3	-42.1	-26.3	4.8
Spain Equity	0.0	-45.6	-63.1	-42.9	-35.0	-52.7	-34.0	4.5
RUSSELL 1000 GROWTH INDEX	0.0	-21.0	-29.5	-18.7	-20.6	-22.9	-13.3	4.3
AUD	7.7	-36.6	-43.5	-42.3	-23.3	-46.6	-27.5	4.3
Jet Kerosene Swap Future	0.0	-25.0	-24.8	-20.5	-21.4	-37.9	-20.1	4.1
US Equity (S&P 500)	0.0	-21.5	-28.9	-20.3	-21.4	-25.4	-11.6	4.0
US Bond (7-10 GOV TR)	7.7	4.9	16.7	11.1	-20.5	19.4	-2.3	0.4
US TIPS INDEX	0.0	-3.6	5.5	-7.5	-16.8	-13.7	14.4	0.3
Japan Bond (10 GOV TR)	0.0	0.3	1.1	1.6	-1.8	2.2	-1.4	0.2
Corporate AAA	0.0	-0.4	1.2	-1.3	-4.4	2.4	0.3	0.2
Italy Bond (10 GOV TR)	0.0	0.5	7.4	-4.0	5.4	-6.4	0.0	0.1
Spain Bond (10 GOV TR)	0.0	0.3	8.7	-1.8	4.5	-8.1	-1.5	0.1

CONCLUSION

Traditional risk budgeting process relies in a large part on a specific decomposition of Value-at-Risk or a Tracking Error. We show that this decomposition possesses an important property of having the highest conditional density on a loss hyperplane that corresponds to a given VaR level. However, focusing on the most likely outcome is leaving out important information which would surely modify the decision making if it was available. We show that a specific Reverse Stress Testing algorithm can be applied to the same risk model to significantly expand the foundation risk decomposition. The scenarios are chosen to satisfy the criteria of plausibility and variety, in other words, to expand the useful information set.

Reverse Stress Testing algorithm has a number of other uses beyond augmenting the standard risk decomposition and budgeting procedures. It can be used to significantly expand the hedging universe and to find "put options" on a specific portfolio in the marketplace without resorting to the expensive OTC hedges.

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